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PARAMETER ESTIMATION OF EXPONENTIATED GENERALIZED INVERTED EXPONENTIAL DISTRIBUTION VIA BAYESIAN APPROACH

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Abstract: In this paper, exponentiated generalized inverted exponential distribution is considered for Bayesian analysis. The expressions for Bayes estimators of the parameter have been derived under squared error, precautionary, entropy, K-loss, and Al-Bayyati's loss functions by using quasi and gamma priors..

Keywords: Bayesian method, exponentiated generalized inverted exponential distribution, quasi and gamma priors, squared error, precautionary, entropy, K-loss, and Al-Bayyati's loss functions.

1. Introduction

The exponentiated generalized inverted exponential distribution (EGIED) was introduced by Oguntunde *et al.* [1]. They obtained the statistical properties of this distribution. The model is positively skewed, its shape could be decreasing or unimodal (depending on the values of the parameters) and it has an inverted bathtub failure rate. The generalized inverse exponential distribution and the inverse exponential distribution are found to be sub-models of this model. The use of this model in situations where the risk is low at the initial stage, increases with time and then decreases (for example breast cancer, bladder cancer). The probability density function of EGIED is given by

$$f(x;\theta) = c\lambda\theta x^{-2} e^{-(\lambda/x)}[1-e^{-(\lambda/x)}]^{c-1}[1-(1-e^{-(\lambda/x)})]^{c\theta-1}; x>0, \tag{1}$$

where c and θ are the shape parameters and λ is the scale parameter.

The joint density function or likelihood function of (1) is given by

$$f(\underline{x};\theta) = (c\lambda\theta)^n \left(\prod_{i=1}^{n} x_i^{-2} e^{-(\lambda/x_i)}\left[1-e^{-(\lambda/x_i)}\right]^{c-1}\right)\exp\left[(\theta-1)\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^{c}\right]\right] \tag{2}$$

The log likelihood function is given by

$$\log f(\underline{x};\theta) = n\log(c\lambda\theta) + \log\left(\prod_{i=1}^{n} x_i^{-2} e^{-(\lambda/x_i)}\left[1-e^{-(\lambda/x_i)}\right]^{c-1}\right) + (\theta-1)\sum_{i=1}^{n}\log\left[1-(1-e^{-(\lambda/x_i)})^c\right] \quad (3)$$

Differentiating (3) with respect to θ and equating to zero, we get the maximum likelihood estimator of θ which is given as

$$\hat{\theta} = n\left(\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}\right)^{-1}. \quad (4)$$

2. Bayesian Method of Estimation

The Bayesian inference procedures have been developed generally under squared error loss function

$$L(\hat{\theta},\theta) = (\hat{\theta},\theta)^2. \quad (5)$$

The Bayes estimator under the above loss function, say, $\hat{\theta}_s$ is the posterior mean, *i.e.*,

$$\hat{\theta}_s = E(\theta) \quad (6)$$

Zellner [2], Basu and Ebrahimi [3] have recognized that the inappropriateness of using symmetric loss function. Norstrom [4] introduced precautionary loss function is given as

$$L(\hat{\theta},\theta) = \frac{(\hat{\theta},\theta)^2}{\hat{\theta}}. \quad (7)$$

The Bayes estimator under this loss function is denoted by $\hat{\theta}_P$ and is obtained as

$$\hat{\theta}_P = \left[E\left(\theta^2\right)\right]^{1/2} \quad (8)$$

Calabria and Pulcini [5] points out that a useful asymmetric loss function is the entropy loss

$$L(\theta) \propto [\delta^p - p\log_e(\delta) - 1]$$

where $\delta = \frac{\hat{\theta}}{\theta}$, and whose minimum occurs at $\hat{\theta} = \theta$. Also, the loss function $L(\delta)$ has been used in Dey *et al*. [6] and Dey and Liu [7], in the original form having $p = 1$. Thus $L(\delta)$ can written be as

$$L(\delta) = b[\delta - \log_e(\delta) - 1];\ b > 0. \quad (9)$$

The Bayes estimator under entropy loss function is denoted by and is obtained by solving the following equation

$$\hat{\theta}_E = \left[E\left(\frac{1}{\theta}\right)\right]^{-1}. \quad (10)$$

Wasan [8] proposed the K-loss function which is given as

$$L(\hat{\theta}, \theta) = \frac{(\hat{\theta} - \theta)^2}{\hat{\theta}\,\theta}. \tag{11}$$

Under K-loss function the Bayes estimator of θ is denoted by $\hat{\theta}_K$ and is obtained as

$$\hat{\theta}_K = \left[\frac{E(\theta)}{E(1/\theta)}\right]^{\frac{1}{2}}. \tag{12}$$

Al-Bayyati [9] introduced a new loss function which is given as

$$L(\hat{\theta}, \theta) = \theta^c (\hat{\theta} - \theta)^2. \tag{13}$$

Under Al-Bayyati's loss function the Bayes estimator of θ is denoted by $\hat{\theta}_{Al}$ and is obtained as

$$\hat{\theta}_{Al} = \frac{E(\theta^{c+1})}{E(\theta^c)}. \tag{14}$$

Let us consider two prior distributions of θ to obtain the Bayes estimators.

(i) **Quasi-prior:** For the situation where we have no prior information about the parameter θ, we may use the quasi density as given by

$$g_1(\theta) = \frac{1}{\theta^d};\ \theta > 0,\ \ d \geq 0, \tag{15}$$

where $d = 0$ leads to a diffuse prior and $d = 1$, a non-informative prior.

(ii) **Gamma prior:** Generally, the gamma density is used as prior distribution of the parameter θ given by

$$g_2(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta};\ \theta > 0. \tag{16}$$

3. Posterior Density Under $g_1(\theta)$

The posterior density of θ under $g_1(\theta)$, on using (2), is given by

$$f(\theta/\underline{x}) = \frac{\left[(c\lambda\theta)^n \left(\prod_{i=1}^{n} x_i^{-2} e^{-(\lambda/x_i)} \left[1 - e^{-(\lambda/x_i)}\right]^{c-1}\right) \exp\left[(\theta - 1)\sum_{i=1}^{n} \log\left[1 - \left(1 - e^{-(\lambda/x_i)}\right)^c\right]\right] \theta^{-d}\right]}{\int_0^\infty \left[(c\lambda\theta)^n \left(\prod_{i=1}^{n} x_i^{-2} e^{-(\lambda/x_i)} \left[1 - e^{-(\lambda/x_i)}\right]^{c-1}\right) \exp\left[(\theta - 1)\sum_{i=1}^{n} \log\left[1 - \left(1 - e^{-(\lambda/x_i)}\right)^c\right]\right] \theta^{-d}\right] d\theta}$$

$$= \frac{\theta^{n-d} e^{-\theta \sum_{i=1}^{n} \log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}}}{\int_0^\infty \theta^{n-d} e^{-\theta \sum_{i=1}^{n} \log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}} d\theta} = \frac{\left(\sum_{i=1}^{n} \log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}\right)^{n-d+1}}{\Gamma(n-d+1)} \theta^{n-d} e^{-\theta \sum_{i=1}^{n} \log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}} \quad (17)$$

Theorem 1. On using (17), we have

$$E\left(\theta^c\right) = \frac{\Gamma(n-d+c+1)}{\Gamma(n-d+1)} \left(\sum_{i=1}^{n} \log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}\right)^{-c}. \quad (18)$$

Proof. By definition,

$$E(\theta^c) = \int \theta^c f\left(\theta/\underline{x}\right) d\theta = \frac{\left(\sum_{i=1}^{n} \log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}\right)^{n-d+1}}{\Gamma(n-d+1)} \int_0^\infty \theta^{(n-d+c)} e^{-\theta \sum_{i=1}^{n} \log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}} d\theta$$

$$= \frac{\left(\sum_{i=1}^{n} \log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}\right)^{n-d+1}}{\Gamma(n-d+1)} \frac{\Gamma(n-d+c+1)}{\left(\sum_{i=1}^{n} \log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}\right)^{n-d+c+1}}$$

$$= \frac{\Gamma(n-d+c+1)}{\Gamma(n-d+1)} \left(\sum_{i=1}^{n} \log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}\right)^{-c}.$$

From equation (18), for c = 1, we have

$$E(\theta) = (n-d+1) \left(\sum_{i=1}^{n} \log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}\right)^{-1}. \quad (19)$$

From equation (18), for c = 2, we have

$$E\left(\theta^2\right) = \left[(n-d+2)(n-d+1)\right] \left[\sum_{i=1}^{n} \log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}\right]^{-2}. \quad (20)$$

From equation (18), for $c = -1$, we have

$$E\left(\frac{1}{\theta}\right)=\frac{1}{(n-d)}\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^{c}\right]^{-1}. \tag{21}$$

From equation (18), for $c = c + 1$, we have

$$E\left(\theta^{c+1}\right)=\frac{\Gamma(n-d+c+2)}{\Gamma(n-d+1)}\left(\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^{c}\right]^{-1}\right)^{-(c+1)}. \tag{22}$$

4. Bayes Estimators Under $g_1(\theta)$

From equation (6), on using (19), the Bayes estimator of θ under squared error loss function is given by

$$\hat{\theta}_S=(n-d+1)\left(\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^{c}\right]^{-1}\right)^{-1}. \tag{23}$$

From equation (8), on using (20), the Bayes estimator of θ under precautionary loss function is obtained as

$$\hat{\theta}_P=\left[(n-d+2)(n-d+1)\right]^{\frac{1}{2}}\left(\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^{c}\right]^{-1}\right)^{-1}. \tag{24}$$

From equation (10), on using (21), the Bayes estimator of θ under entropy loss function is given by

$$\hat{\theta}_E=(n-d)\left(\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^{c}\right]^{-1}\right)^{-1}. \tag{25}$$

From equation (12), on using (19) and (21), the Bayes estimator of θ under K-loss function is given by

$$\hat{\theta}_K=\left[(n-d+1)(n-d)\right]^{\frac{1}{2}}\left(\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^{c}\right]^{-1}\right)^{-1}. \tag{26}$$

From equation (14), on using (18) and (22), the Bayes estimator of θ under Al-Bayyati's loss function comes out to be

$$\hat{\theta}_{Al}=(n-d+c+1)\left(\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^{c}\right]^{-1}\right)^{-1}. \tag{27}$$

5. Posterior Density Under $g_2(\theta)$

Under $g_2(\theta)$, the posterior density of è, using equation (2), is obtained as

$$f\left(\theta/\underline{x}\right)=\frac{\left[\begin{array}{l}\left(c\lambda\theta\right)^n\left(\prod_{i=1}^{n}x_i^{-2}e^{-(\lambda/x_i)}\left[1-e^{-(\lambda/x_i)}\right]^{c-1}\right)\\ \qquad\exp\left[\left(\theta-1\right)\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]\right]\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}\end{array}\right]}{\int_0^\infty\left[\begin{array}{l}\left(c\lambda\theta\right)^n\left(\prod_{i=1}^{n}x_i^{-2}e^{-(\lambda/x_i)}\left[1-e^{-(\lambda/x_i)}\right]^{c-1}\right)\\ \qquad\exp\left[\left(\theta-1\right)\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]\right]\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}\end{array}\right]d\theta}$$

$$=\frac{\theta^{n+\alpha-1}\,e^{-\left(\beta+\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}\right)\theta}}{\int_0^\infty\theta^{n+\alpha-1}\,e^{-\left(\beta+\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}\right)\theta}d\theta}=\frac{\theta^{n+\alpha-1}\,e^{-\left(\beta+\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}\right)\theta}}{\Gamma\left(n+\alpha\right)\Big/\left(\beta+\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}\right)^{n+\alpha}}$$

$$=\frac{\left(\beta+\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}\right)^{n+\alpha}}{\Gamma\left(n+\alpha\right)}\theta^{n+\alpha-1}\,e^{-\left(\beta+\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}\right)\theta} \tag{28}$$

Theorem 2. On using (28), we have

$$E\left(\theta^c\right)=\frac{\Gamma\left(n+\alpha+c\right)}{\Gamma\left(n+\alpha\right)}\left(\beta+\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}\right)^{-c}. \tag{29}$$

Proof. By definition,

$$E\left(\theta^c\right)=\int\theta^c f\left(\theta/\underline{x}\right)d\theta$$

$$=\frac{\left(\beta+\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}\right)^{n+\alpha}}{\Gamma\left(n+\alpha\right)}\int_0^\infty\theta^{n+\alpha+c-1}\,e^{-\left(\beta+\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}\right)\theta}d\theta$$

$$=\left(\left(\beta+\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}\right)^{n+\alpha}\Big/\Gamma(n+\alpha)\right)$$

$$\times\left(\Gamma(n+\alpha+c)\Big/\left(\beta+\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}\right)^{n+\alpha+c}\right)$$

$$=\frac{\Gamma(n+\alpha+c)}{\Gamma(n+\alpha)}\left(\beta+\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}\right)^{-c}$$

From equation (29), for $c = 1$, we have

$$E(\theta)=(n+\alpha)\left(\beta+\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}\right)^{-1}. \tag{30}$$

From equation (29), for $c = 2$, we have

$$E(\theta^2)=\left[(n+\alpha+1)(n+\alpha)\right]\left(\beta+\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}\right)^{-2}. \tag{31}$$

From equation (29), for $c = -1$, we have

$$E\left(\frac{1}{\theta}\right)=\frac{1}{(n+\alpha-1)}\left(\beta+\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}\right). \tag{32}$$

From equation (29), for $c = c + 1$, we have

$$E(\theta^{c+1})=\frac{\Gamma(n+\alpha+c+1)}{\Gamma(n+\alpha)}\left(\beta+\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}\right)^{-(c+1)}. \tag{33}$$

6. Bayes Estimators Under $g_2(\theta)$

From equation (6), on using (30), the Bayes estimator of θ under squared error loss function is given by

$$\hat{\theta}_S=(n+\alpha)\left(\beta+\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}\right)^{-1}. \tag{34}$$

From equation (8), on using (31), the Bayes estimator of θ under precautionary loss function is obtained as

$$\hat{\theta}_P = \left[(n+\alpha+1)(n+\alpha)\right]^{\frac{1}{2}} \left(\beta + \sum_{i=1}^{n} \log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}\right)^{-1}. \tag{35}$$

From equation (10), on using (32), the Bayes estimator of θ under entropy loss function is given by

$$\hat{\theta}_E = (n+\alpha+1)\left(\beta + \sum_{i=1}^{n} \log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}\right)^{-1}. \tag{36}$$

From equation (12), on using (30) and (32), the Bayes estimator of θ under K-loss function is given by

$$\hat{\theta}_K = \left[(n+\alpha)(n+\alpha-1)\right]^{\frac{1}{2}} \left(\beta + \sum_{i=1}^{n} \log\left[1-\left(1-e^{-(\lambda/x_i)}\right)^c\right]^{-1}\right)^{-1}. \tag{37}$$

From equation (14), on using (29) and (33), the Bayes estimator of θ under Al-Bayyati's loss function comes out to be

$$\hat{\theta}_{Al} = (n+\alpha+c)\left(\beta + \sum_{i=1}^{n} \log\left[1-\left(1-\mathrm{e}^{-(\lambda/x_i)}\right)^c\right]^{-1}\right)^{-1}. \tag{38}$$

Conclusion

In this paper, we have obtained a number of estimators of parameter of exponentiated generalized inverted exponential distribution. In equation (23), (24), (25), (26) and (27) we have obtained the Bayes estimators under different loss functions using quasi prior. In equation (34), (35), (36), (37) and (38) we have obtained the Bayes estimators under different loss functions using gamma prior. In the above equation, it is clear that the Bayes estimators depend upon the parameters of the prior distribution. We therefore recommend that the estimator's choice lies according to the value of the prior distribution which in turn depends on the situation at hand.

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